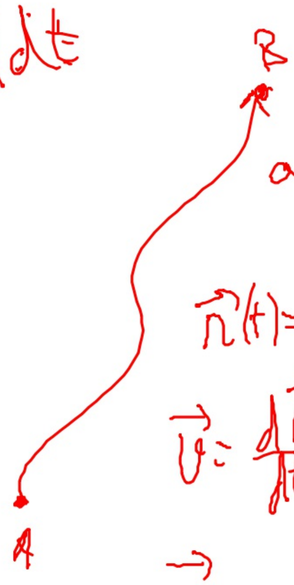
$$\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot \frac{\frac{d\vec{r}}{dt}}{\cancel{|\vec{v}|}}\,\cancel{|\vec{v}|}\,dt$$

$$= \int_a^b \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt$$



$a \le t \le b$

$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{v} = \dfrac{d\vec{r}}{dt}$

$\vec{T} = \dfrac{\vec{v}}{|\vec{v}|}$

Vector Fields:

$\overrightarrow{\nabla f} = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$

examples: $f(x, y)$

or

$f(x, y, z)$

$\vec{F} = \overrightarrow{\nabla f}$ is a vector field.

f potential function

$\vec{F}$ = Force field.

$\vec{T}$ unit tangent vector

<u>Work</u> $\int_C \vec{F}\cdot\vec{T}\,ds$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$

$$= \int_a^b \vec{F}\cdot\frac{\vec{v}}{\cancel{|\vec{v}|}}\,\cancel{|\vec{v}|}\,dt$$

$$= \int_a^b \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt$$

$\vec{v} = \frac{d\vec{r}}{dt}$

$a \le t \le b$

7-b

$\vec{F} = 3y\,\vec{i} + 2x\,\vec{j} + 4z\,\vec{k}$

$\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + t^4\,\vec{k} \quad 0 \le t \le 1$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^1 \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_0^1 (16t^7 + 7t^2)\,dt$$

$\vec{F} = 3t^2\,\vec{i} + 2t\,\vec{j} + 4t^4\,\vec{k}$

$\frac{d\vec{r}}{dt} = \vec{i} + 2t\,\vec{j} + 4t^3\,\vec{k}$

$\vec{F}\cdot\frac{d\vec{r}}{dt} = 3t^2 + 4t^2 + 16t^7$

$$\left(2t^8 + \frac{7}{3}t^3\right)\Big|_0^1 = 2 + \frac{7}{3} = \frac{13}{3}.$$

27

$\vec{F} = xy\,\vec{i} + (y-x)\,\vec{j}$

$(2t^2+3t+1) = M = xy = (t+1)(2t+1)$; $dy = 2\,dt$

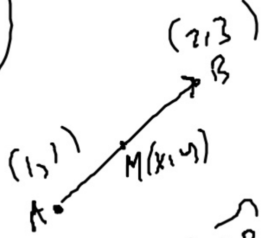
$N = y - x = t$; $dx = dt$

Work $= \int_C M\,dx + N\,dy$

$= \int_0^1 (2t^2+3t+1+2t)\,dt$

$= \dots = \frac{25}{6}$??

$\vec{R}(t) = (t+1)\vec{i} + (2t+1)\vec{j}$.



$\vec{AM} = t\,\vec{AB}$

$0 \le t \le 1$

$\vec{AB} = \left|\begin{matrix}1\\2\end{matrix}\right.$

$\vec{AM} = \left|\begin{matrix}x-1\\y-1\end{matrix}\right.$

$x-1 = t$

$y-1 = 2t$

$x = t+1$

$y = 2t+1$

28) ★

$f(x,y) = (x+y)^2$

$\vec{F} = \vec{\nabla} f = \underbrace{2(x+y)}_{M}\,\vec{i} + \underbrace{2(x+y)}_{N}\,\vec{j}$

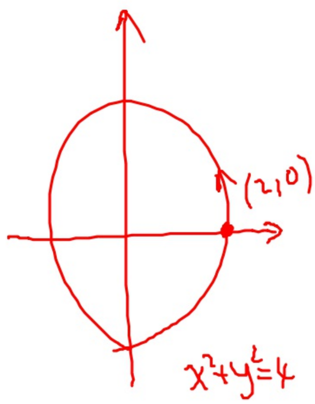
$M = N = 2(2\sin t + 2\cos t)$

$$\text{work} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} M\,dx + N\,dy$$

$$= \int_0^{2\pi} \Big(4(\sin t + \cos t)(-2\sin t) + 4(\sin t + \cos t)(2\cos t)\Big)\,dt$$

$$= \int_0^{2\pi} \left(-8\sin^2 t + 8\cos^2 t\right) dt$$

$$= 8\int_0^{2\pi} \cos 2t\,dt = \Big[4\sin 2t\Big]_0^{2\pi} = 0$$



$x = 2\cos t$

$y = 2\sin t$

$0 \le t \le 2\pi$

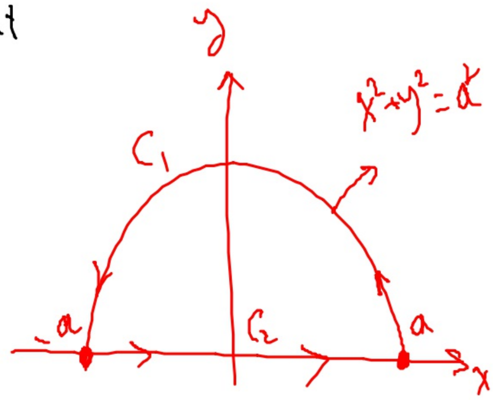
31) Circulation $= \int_C M\,dx + N\,dy = \int_{C_1} \cdots + \int_{C_2} \cdots = 0$

$$\boxed{\vec{F} = x\vec{i} + y\vec{j}}$$

$$\int_{C_1} M\,dx + N\,dy = \int_0^{\pi} \left(a\cos t\,(-a\sin t) + a\sin t\; a\cos t\right) dt$$

$$= \int_0^{\pi} 0\,dt = 0$$

$$\int_{C_2} M\,dx + N\,dy = \int_{-a}^{a} t\,dt = 0$$



$C_1: \vec{r}_1(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

$0 \le t \le \pi$

$C_2: \vec{r}_2(t) = t\vec{i}$; $-a \le t \le a$

$$\underline{\text{Flux}} = \int_C M\,dy - N\,dx = \int_{C_1} \cdots + \int_{C_2} \cdots = \pi a^2$$

$\vec{F} = x\vec{i} + y\vec{j}$

$$\int_{C_1} M\,dy - N\,dx = \int_0^{\pi} (a^2\cos^2 t + a^2 \sin^2 t)\,dt = \int_0^{\pi} a^2\,dt = \pi a^2.$$

$$\int_{C_2} M\,dy - N\,dx = \int_{-a}^{a} (0+0)\,dt = 0.$$

53)

$\vec{r}(t) = t\vec{i} + t^2\vec{j} + t\vec{k}$; $0 \le t \le 1$

$\frac{d\vec{r}}{dt} = \vec{i} + 2t\vec{j} + \vec{k}$.

$\boxed{\vec{F} = xy\vec{i} + y\vec{j} - yz\vec{k}} = t^3\vec{i} + t^2\vec{j} - t^3\vec{k}$.

$\text{Flow} = \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt = \int_0^1 (t^3 + 2t^3 - t^3)\, dt$

$= \int_0^1 2t^3\, dt = \frac{1}{2}$.

C is the intersection of $y = x^2$ & $z = x$

let $x = t$ "Hint"

Conservative fields : $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$.

If $\vec{F} = \overrightarrow{\nabla f}$.

$M = f_x$

$N = f_y$

$P = f_z$

$$f_{xy} = \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} = f_{yx}$$

$$f_{xz} = \boxed{\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}} = f_{zx}$$

$$f_{yz} = \boxed{\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}} = f_{zy}$$

ex3 $\vec{F} = (e^x \cos y + yz)\vec{i} + (xz - e^x \sin y)\vec{j} + (xy + z)\vec{k}$. Conservative ✓. ???

Finding f: $M = f_x \Rightarrow f_x = e^x \cos y + yz$

$\Rightarrow f(x,y,z) = \int f_x \, dx = e^x \cos y + xyz + g(y,z)$ g is const w.r. to x.

$xz - e^x \sin y = -e^x \sin y + xz + \frac{\partial g}{\partial y} \Longrightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$

Next: $N = f_y \Rightarrow$

$\therefore f(x,y,z) = e^x \cos y + xyz + h(z)$

Now $P = f_z \Rightarrow xy + z = xy + h'(z) \Rightarrow h'(z) = z \Rightarrow h(z) = \frac{z^2}{2} + C$ $C \in \mathbb{R}$

$\therefore f(x,y,z) = e^x \cos y + xyz + \frac{z^2}{2} + C$

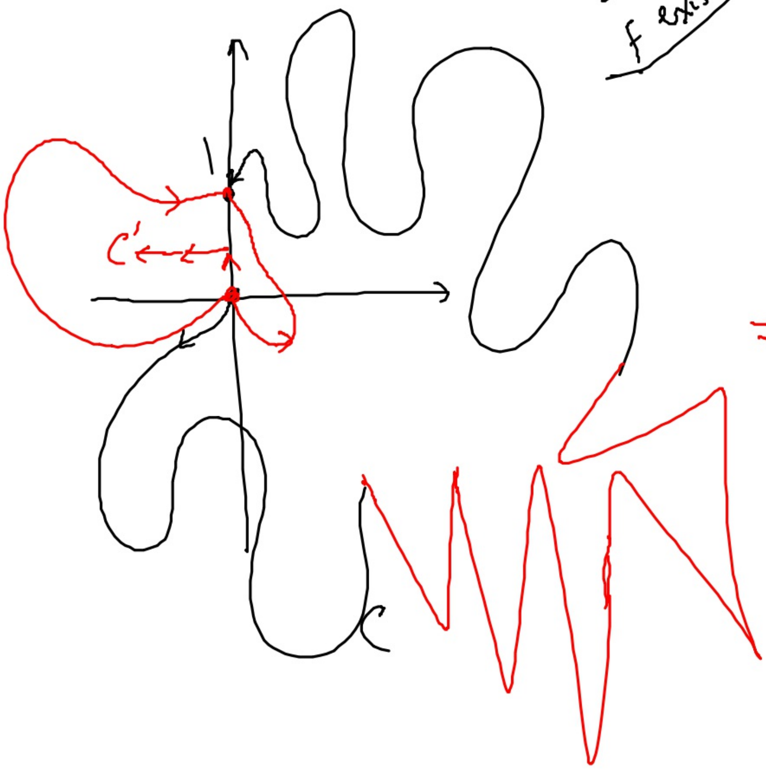
part c

$$\int_{(0,\pi,1)}^{(0,\frac{\pi}{2},1)} \vec{F} \cdot d\vec{r} = f(0,\tfrac{\pi}{2},1) - f(0,\pi,1) = \frac{1}{2} + C - \left(-1 + \frac{1}{2} + C\right)$$

$$= 1.$$

$\vec{F}$ cons

f exists



$$\int_C \vec{F}\cdot d\vec{r} \cancel{=} f(1,0) - f(0,0)$$

$$= \int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r} \,.$$

\# 5 $\vec{F} = \underbrace{(z+y)}_{M}\vec{i} + \underbrace{z}_{N}\vec{j} + \underbrace{(y+x)}_{P}\vec{k}$

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 0$$

$\therefore$ $\vec{F}$ is not cons.

6) $\vec{F} = \underbrace{e^x \cos y}_{M}\, \vec{i} - \underbrace{e^x \sin y}_{N}\, \vec{j} + \underbrace{z}_{P}\, \vec{k}$.

Finding $f(x,y,z)$ (potential function)

- $M = f_x \Rightarrow f(x,y,z) = \int f_x\, dx = \int e^x \cos y\, dx = e^x \cos y + g(y,z)$
- $N = f_y \Rightarrow -e^x \sin y = -e^x \sin y + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow$

$g(y,z) = h(z)$

$\therefore f(x,y,z) = e^x \cos y + h(z)$

- $P = f_z \Rightarrow z = h'(z) \Rightarrow h(z) = \frac{z^2}{2} + C$

$\therefore f(x,y,z) = e^x \cos y + \frac{z^2}{2} + C$.

$\frac{\partial M}{\partial y} = -e^x \sin y = \frac{\partial N}{\partial x}$

$\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$ ✓

$\frac{\partial N}{\partial z} = 0 = \frac{\partial P}{\partial y}$ ✓

$\therefore F$ is cons.

Vector field : $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ $\Rightarrow$ conservative $\vec{F} = \nabla f$.

$\Updownarrow$

line integral : $\int_A^B \vec{F}\cdot d\vec{r} = f(B) - f(A)$ $\Rightarrow$ Path ind

$\exists$ $f(x,y,z)$
potential

$\Leftrightarrow$ diff. form
is exact

#17 $I = \int_{(1,0,0)}^{(0,1,1)} \underbrace{\sin y \cos x}_{M}\, dx + \underbrace{\cos y \sin x}_{N}\, dy + \underbrace{dz}_{P=1}$ is exact ??

Finding $f(x,y,z)$:

- $f(x,y,z) = \int \sin y \cos x \, dx = \sin y \sin x + g(y,z)$
- $N = f_y \Rightarrow \cos y \sin x = \cos y \sin x + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$

  and $f(x,y,z) = \sin y \sin x + h(z)$
- $P = f_z \Rightarrow 1 = h'(z) = h(z) = z + C$

  $\therefore f(x,y,z) = \sin x \sin y + z + C$

$\therefore I = f(0,1,1) - f(1,0,0) = 1 + C - C = 1$

#18

$$I = \int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)dy + \frac{1}{z}\,dz$$

is exact ?

$y>0$

Finding potential function: $f(x,y,z) = \int 2\cos y\,dx = 2x\cos y + g(y,z)$

$N = f_y \Rightarrow \frac{1}{y} - 2x\sin y = -2x\sin y + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y} \Rightarrow g = \ln y + h(z)$

$\therefore f(x,y,z) = 2x\cos y + \ln y + h(z)$

$P = f_z \Rightarrow \frac{1}{z} = h'(z) \Rightarrow h(z) = \ln z + C$

$\therefore f(x,y,z) = 2x\cos y + \ln y + \ln z + C$

$\therefore I = f\left(1,\frac{\pi}{2},2\right) - f(0,2,1)$